# Proof of proposition 2

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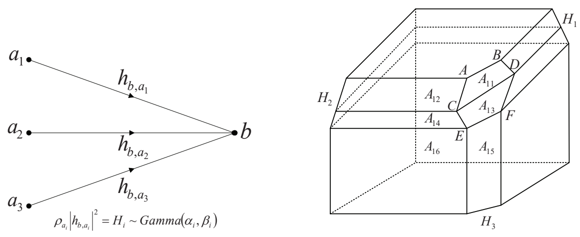



(a) MAC channel (b) rate regions of $\tilde{P}_{123}$

Fig. 1. 3-user MAC channel and rate regions

The considered 3-user MAC channel is displayed in Fig. 1 (a), user $a_i$ ($i \in \{1,2,3\}$) transmit message with rate $R$ using power $\rho_{a_i}$, denote $\eta_i = 2^{iR} - 1$. Letting $H_i = \rho_{a_i}|h_{b,a_i}|^2$ for $i \in \{1,2,3\}$ leads to $H_i \sim \text{Gamma}(\alpha_i, \beta_i)$, where $\beta_i = \frac{\alpha_i}{\Omega_{b,a_i}\rho_{a_i}}$. Then the probability density function (PDF) of $H_i$ is [1]

$$f_X(x) = \frac{(\beta_i)^{\alpha_i} x^{\alpha_i - 1} e^{-\beta_i x}}{\Gamma(\alpha_i)}, x \geq \frac{1}{2}, i \in \{1,2,3\}, \tag{1}$$

in which $\Gamma(\alpha_i) = (\alpha_i - 1)!$. The associated cumulative distribution function (CDF) is [2]

$$F(x;\alpha_i,\beta_i) = 1 - \sum_{j=0}^{\alpha_i - 1} \frac{(\beta_i x)^j}{j!} e^{-\beta_i x}, i \in \{1,2,3\}. \tag{2}$$

The probability that the receiver $b$ can correctly decode all transmitters $a_1$, $a_2$ and $a_3$ is denoted by $\tilde{P}_{123}$. Using the definitions of *individual error probability* [3], $\tilde{P}_{123}$ can be calculated as $\tilde{P}_{123} = \Pr\{E_{1,\mathcal{A}}, E_{2,\bar{\mathcal{A}}} | \mathcal{A} = \{a_1,a_2,a_3\}, \mathcal{L} = \{a_1,a_2,a_3\}\}$, i.e.

$$\begin{aligned}\tilde{P}_{123} =& \Pr\big\{\log_2(1+H_i) > R, \log_2(1+H_i+H_j) > 2R, \\ & \log_2(1+H_1+H_2+H_3) > 3R\big\} \\ =& \Pr\big\{H_i > 2^R - 1, H_i + H_j > 2^{2R} - 1, \\ & H_1 + H_2 + H_3 > 2^{3R} - 1, \forall i,j \in \{1,2,3\}, i \neq j\big\}.\end{aligned}$$

The integration region can be divided into six non-overlapping sub-regions as Fig. 1 (b), denoted by $A_{11}$ to $A_{16}$ respectively. The coordinates of point $A \sim F$ are respectively $A(\eta_3 - \eta_2, \eta_2 - \eta_1, \eta_1)$, $B(\eta_2 - \eta_1, \eta_3 - \eta_2, \eta_1)$, $C(\eta_3 - \eta_2, \eta_1, \eta_2 - \eta_1)$, $D(\eta_1, \eta_3 - \eta_2, \eta_2 - \eta_1)$, $E(\eta_2 - \eta_1, \eta_1, \eta_3 - \eta_2)$, $F(\eta_1, \eta_2 - \eta_1, \eta_3 - \eta_2)$. The regions of $A_{11}$ to $A_{16}$ are three-dimensional which consist of sectional area ($H_2$, $H_3$) and height ($H_1$). To represent simply and clearly, we use some signals to replace long expressions. Here we denote $\sigma_1 = \alpha_2 + i - j$, specifically, we have

$$\begin{aligned}\mathcal{P}\{A_{11}\} =& \int_{\eta_1}^{\eta_2 - \eta_1} \int_{\eta_2 - x_3}^{\eta_3 - \eta_2} \int_{\eta_3 - x_2 - x_3}^{\infty} f_{H_1|x_2,x_3}(x_1) \\ & \cdot f_{H_2|x_3}(x_2) f_{H_3}(x_3) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \\ =& \int_{\eta_1}^{\eta_2 - \eta_1} \int_{\eta_2 - x_3}^{\eta_3 - \eta_2} (1 - F(\eta_3 - x_2 - x_3; \alpha_1, \beta_1)) \\ & \cdot \frac{(\beta_2)^{\alpha_2} x_2^{\alpha_2 - 1} e^{-\beta_2 x_2}}{\Gamma(\alpha_2)} \mathrm{d}x_2 f_{H_3}(x_3) \mathrm{d}x_3 \\ =& \int_{\eta_1}^{\eta_2 - \eta_1} \int_{\eta_2 - x_3}^{\eta_3 - \eta_2} \frac{\beta_2^{\alpha_2} e^{-\beta_1(\eta_3 - x_3)}}{\Gamma(\alpha_2)} \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{i} \binom{i}{j} \frac{(-1)^{i-j}}{i!} \\ & \cdot \beta_1^i (\eta_3 - x_3)^j x_2^{\sigma_1 - 1} e^{-(\beta_2 - \beta_1) x_2} \mathrm{d}x_2 f_{H_3}(x_3) \mathrm{d}x_3.\end{aligned}$$

When $\beta_1 = \beta_2$, then the above expression is simply

$$\begin{aligned}\mathcal{P}\{A_{11}\} =& \int_{\eta_1}^{\eta_2 - \eta_1} \frac{\beta_2^{\alpha_2} e^{-\beta_1(\eta_3 - x_3)}}{\Gamma(\alpha_2)} \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{i} \binom{i}{j} \frac{\beta_1^i (-1)^{i-j}}{i! \sigma_1} \\ & \cdot (\eta_3 - x_3)^j \left((\eta_3 - \eta_2)^{\sigma_1} - (\eta_2 - x_3)^{\sigma_1}\right) f_{H_3}(x_3) \mathrm{d}x_3 \\ =& \mathcal{P}\{A_{11}^1\} - \mathcal{P}\{A_{11}^2\},\end{aligned} \tag{3}$$

$$\begin{aligned}\mathcal{P}\{A_{11}^1\} =& \int_{\eta_1}^{\eta_2 - \eta_1} \frac{\beta_2^{\alpha_2} e^{-\beta_1(\eta_3 - x_3)}}{\Gamma(\alpha_2)} \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{i} \binom{i}{j} \frac{\beta_1^i (-1)^{i-j}}{i! \sigma_1} \\ & \cdot (\eta_3 - x_3)^j (\eta_3 - \eta_2)^{\sigma_1} \frac{\beta_3^{\alpha_3} x_3^{\alpha_3 - 1} e^{-\beta_3 x_3}}{\Gamma(\alpha_3)} \mathrm{d}x_3 \\ =& \int_{\eta_1}^{\eta_2 - \eta_1} \frac{\beta_2^{\alpha_2} \beta_3^{\alpha_3} e^{-\beta_1 \eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)} \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{i} \binom{i}{j} \frac{\beta_1^i (-1)^{i-j}}{i! \sigma_1} \\ & \cdot \sum_{l=0}^{j} \binom{j}{l} (-1)^{j-l} \eta_3^l (\eta_3 - \eta_2)^{\sigma_1} x_3^{\sigma_2 - 1} e^{-(\beta_3 - \beta_1) x_3} \mathrm{d}x_3.\end{aligned}$$

in which $\sigma_2 = \alpha_3 + j - l$ and $\beta' = \beta_3 - \beta_1$, if $\beta' = 0$ ($\beta_3 = \beta_1$), we can get $\mathcal{P}\{A_{11}^1\}$ as

$$\begin{aligned}\mathcal{P}\{A_{11}^1\} =& \frac{\beta_2^{\alpha_2} \beta_3^{\alpha_3} e^{-\beta_1 \eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)} \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{i} \sum_{l=0}^{j} \binom{j}{l} \binom{i}{j} \frac{(\beta_1)^i (-1)^{i-l}}{i! \sigma_1 \sigma_2} \\ & \cdot \eta_3^l (\eta_3 - \eta_2)^{\sigma_1} \left((\eta_2 - \eta_1)^{\sigma_2} - \eta_1^{\sigma_2}\right).\end{aligned}$$

if $\beta' \neq 0$ ($\beta_3 \neq \beta_1$), we can get $\mathcal{P}\{A_{11}^1\}$ as

$$\begin{aligned}\mathcal{P}\{A_{11}^1\} =& \frac{\beta_2^{\alpha_2} \beta_3^{\alpha_3} e^{-\beta_1 \eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)} \sum_{i=0}^{\alpha_1 - 1} \sum_{j=0}^{i} \sum_{l=0}^{j} \binom{j}{l} \binom{i}{j} \frac{(\beta_1)^i (-1)^{i-l}}{i! \sigma_1 (\beta')^{\sigma_2}} \\ & \cdot \eta_3^l (\eta_3 - \eta_2)^{\sigma_1} \Gamma(\sigma_2) \left(F(\eta_2 - \eta_1; \sigma_2, \beta') - F(\eta_1; \sigma_2, \beta')\right).\end{aligned}$$

For $\mathcal{P}\{A_{11}^2\}$, when $\beta_1=\beta_2$, we have

$$\mathcal{P}\{A_{11}^2\}=\int_{\eta_1}^{\eta_2-\eta_1}\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i\eta_3^l}{i!\sigma_1}\cdot\sum_{p=0}^{\sigma_1}\binom{\sigma_1}{p}(-1)^{\sigma_1+i-l-p}\eta_2^p x_3^{\sigma_3-1}e^{-\beta' x_3}\mathrm{d}x_3,$$

where $\sigma_3=\alpha_2+\alpha_3+i-l-p$ and $\beta'=\beta_3-\beta_1$. If $\beta'=0$, we can get $\mathcal{P}\{A_{11}^2\}$ as

$$\mathcal{P}\{A_{11}^2\}=\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{p=0}^{\sigma_1}\binom{\sigma_1}{p}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i\eta_3^l}{i!\sigma_1\sigma_3}\cdot(-1)^{\sigma_1+i-l-p}\eta_2^p\left((\eta_2-\eta_1)^{\sigma_3}-\eta_1^{\sigma_3}\right).$$

if $\beta'\neq 0$, we can get $\mathcal{P}\{A_{11}^2\}$ as

$$\mathcal{P}\{A_{11}^2\}=\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{p=0}^{\sigma_1}\binom{\sigma_1}{p}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i\eta_3^l}{i!\sigma_1(\beta')^{\sigma_3}}\cdot(-1)^{\sigma_1+i-l-p}\eta_2^p\Gamma(\sigma_3)\cdot\left(F\left(\eta_2-\eta_1;\sigma_3,\beta'\right)-F\left(\eta_1;\sigma_3,\beta'\right)\right).$$

When $\beta_1$ and $\beta_2$ are not the same, then we can obtain

$$\mathcal{P}\{A_{11}\}=\int_{\eta_1}^{\eta_2-\eta_1}\frac{\beta_2^{\alpha_2}e^{-\beta_1(\eta_3-x_3)}}{\Gamma(\alpha_2)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i(-1)^{i-j}}{i!(\beta_2-\beta_1)^{\sigma_1}}\cdot(\eta_3-x_3)^j\Gamma(\sigma_1)\left(F\left(\eta_3-\eta_2;\sigma_1,\beta_2-\beta_1\right)-F\left(\eta_2-x_3;\sigma_1,\beta_2-\beta_1\right)\right)f_{H_3}(x_3)\mathrm{d}x_3.$$

Set $F'(x;\alpha,\beta)=1-F(x;\alpha,\beta)$, then $\mathcal{P}\{A_{11}\}$ can be expressed as

$$\begin{aligned}\mathcal{P}\{A_{11}\}=&\int_{\eta_1}^{\eta_2-\eta_1}\frac{\beta_2^{\alpha_2}e^{-\beta_1(\eta_3-x_3)}}{\Gamma(\alpha_2)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{(\beta_1)^i(-1)^{i-j}}{i!(\beta_2-\beta_1)^{\sigma_1}}\\&\cdot(\eta_3-x_3)^j\Gamma(\sigma_1)\left(F'\left(\eta_2-x_3;\sigma_1,\beta_2-\beta_1\right)-F'\left(\eta_3-\eta_2;\sigma_1,\beta_2-\beta_1\right)\right)f_{H_3}(x_3)\mathrm{d}x_3\\=&\mathcal{P}\{A_{11}^1\}-\mathcal{P}\{A_{11}^2\}.\end{aligned}\tag{4}$$

$$\mathcal{P}\{A_{11}^1\}=\int_{\eta_1}^{\eta_2-\eta_1}\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i\Gamma(\sigma_1)}{i!(\beta_2-\beta_1)^{\sigma_1}}\cdot\sum_{l=0}^{j}\binom{j}{l}(-1)^{i-l}\eta_3^l e^{-(\beta_2-\beta_1)\eta_2}\sum_{p=0}^{\sigma_1-1}\sum_{q=0}^{p}\binom{p}{q}\cdot\frac{(\beta_2-\beta_1)^p(-1)^{p-q}\eta_2^q}{p!}x_3^{\sigma_2'-1}e^{-(\beta_3-\beta_2)x_3}\mathrm{d}x_3,$$

in which $\sigma_2'=\alpha_3+j-l+p-q$ and $\hat{\beta}'=\beta_3-\beta_2$, if $\hat{\beta}'=0$, $\mathcal{P}\{A_{11}^1\}$ can be calculated as

$$\mathcal{P}\{A_{11}^1\}=\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i\Gamma(\sigma_1)}{i!(\beta_2-\beta_1)^{\sigma_1}}\cdot\sum_{l=0}^{j}\sum_{p=0}^{\sigma_1-1}\sum_{q=0}^{p}\binom{p}{q}\binom{j}{l}\frac{(-1)^{i-l}\eta_3^l e^{-(\beta_2-\beta_1)\eta_2}}{p!\sigma_2'}\cdot(\beta_2-\beta_1)^p(-1)^{p-q}\eta_2^q\left((\eta_2-\eta_1)^{\sigma_2'}-\eta_1^{\sigma_2'}\right).$$

if $\hat{\beta}'\neq 0$, $\mathcal{P}\{A_{11}^1\}$ can be calculated as

$$\mathcal{P}\{A_{11}^1\}=\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i\Gamma(\sigma_1)e^{-(\beta_2-\beta_1)\eta_2}}{i!(\beta_2-\beta_1)^{\sigma_1}}\cdot\sum_{l=0}^{j}\sum_{p=0}^{\sigma_1-1}\sum_{q=0}^{p}\binom{p}{q}\binom{j}{l}\frac{(-1)^{i-l+p-q}\eta_3^l(\beta_2-\beta_1)^p\eta_2^q}{p!(\hat{\beta}')^{\sigma_2'}}\cdot\Gamma(\sigma_2')\left(F\left(\eta_2-\eta_1;\sigma_2',\hat{\beta}'\right)-F\left(\eta_1;\sigma_2',\hat{\beta}'\right)\right).$$

For $\mathcal{P}\{A_{11}^2\}$, when $\beta_1\neq\beta_2$, we have

$$\mathcal{P}\{A_{11}^2\}=\int_{\eta_1}^{\eta_2-\eta_1}\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i\Gamma(\sigma_1)}{i!(\beta_2-\beta_1)^{\sigma_1}}\cdot\sum_{l=0}^{j}\binom{j}{l}(-1)^{i-l}\eta_3^l\left(1-F\left(\eta_3-\eta_2;\sigma_1,\beta_2-\beta_1\right)\right)\cdot x_3^{\sigma_2-1}e^{-(\beta_3-\beta_1)x_3}\mathrm{d}x_3\tag{5}$$

if $\beta'=\beta_3-\beta_1=0$, we get

$$\mathcal{P}\{A_{11}^2\}=\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_2\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i\Gamma(\sigma_1)\eta_3^l}{i!(\beta_2-\beta_1)^{\sigma_1}}\cdot\frac{(-1)^{i-l}}{\sigma_2}F'(\eta_3-\eta_2;\sigma_1,\beta_2-\beta_1)\left((\eta_2-\eta_1)^{\sigma_2}-\eta_1^{\sigma_2}\right).$$

otherwise $\beta'\neq 0$, we get

$$\mathcal{P}\{A_{11}^2\}=\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_2\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i\Gamma(\sigma_1)\eta_3^l}{i!(\beta_2-\beta_1)^{\sigma_1}}\cdot\frac{\Gamma(\sigma_2)(-1)^{i-l}}{(\beta')^{\sigma_2}}\left(1-F\left(\eta_3-\eta_2;\sigma_1,\beta_2-\beta_1\right)\right)\cdot\left(F\left(\eta_2-\eta_1;\sigma_2',\beta'\right)-F\left(\eta_1;\sigma_2,\beta'\right)\right).$$

$$\begin{aligned}\mathcal{P}\{A_{12}\}=&\int_{\eta_1}^{\eta_2-\eta_1}\int_{\eta_3-\eta_2}^{\infty}\int_{\eta_2-x_3}^{\infty}f_{H_1|x_2,x_3}(x_1)\cdot f_{H_2|x_3}(x_2)f_{H_3}(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\=&\int_{\eta_1}^{\eta_2-\eta_1}\int_{\eta_3-\eta_2}^{\infty}\left(1-F\left(\eta_2-x_3;\alpha_1,\beta_1\right)\right)\cdot\frac{\beta_2^{\alpha_2}x_2^{\alpha_2-1}e^{-\beta_2x_2}}{\Gamma(\alpha_2)}\mathrm{d}x_2\cdot f_{H_3}(x_3)\mathrm{d}x_3\\=&\int_{\eta_1}^{\eta_2-\eta_1}\left(1-F\left(\eta_2-x_3;\alpha_1,\beta_1\right)\right)\cdot\left(1-F\left(\eta_3-\eta_2;\alpha_2,\beta_2\right)\right)\cdot f_{H_3}(x_3)\mathrm{d}x_3\\=&\int_{\eta_1}^{\eta_2-\eta_1}\frac{\beta_3^{\alpha_3}e^{-\beta_1\eta_2}}{\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i(-1)^{i-j}\eta_2^j}{i!}\cdot\left(1-F\left(\eta_3-\eta_2;\alpha_2,\beta_2\right)\right)x_3^{\grave{\sigma}_1-1}e^{-(\beta_3-\beta_1)x_3}\mathrm{d}x_3\end{aligned}\tag{6}$$

where $\grave{\sigma}_1=\alpha_3+i-j$. Again there are two kind of conditions, one is $\beta'=\beta_3-\beta_1=0$, then we have

$$\mathcal{P}\{A_{12}\}=\frac{\beta_3^{\alpha_3}e^{-\beta_1\eta_2}}{\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i(-1)^{i-j}\eta_2^j}{i!\grave{\alpha}_1}\cdot\left((\eta_2-\eta_1)^{\grave{\alpha}_1}-\eta_1^{\grave{\alpha}_1}\right)\left(1-F\left(\eta_3-\eta_2;\alpha_2,\beta_2\right)\right),$$

the other one is when $\beta_3$ not equal to $\beta_1$, we have

$$\mathcal{P}\{A_{12}\}=\frac{\beta_3^{\alpha_3}e^{-\beta_1\eta_2}}{\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i(-1)^{i-j}\eta_2^j\Gamma(\grave{\sigma}_1)}{i!(\beta')^{\grave{\sigma}_1}} \cdot\left(F(\eta_2-\eta_1;\grave{\sigma}_1,\beta')-F(\eta_1;\grave{\sigma}_1,\beta')\right)F'(\eta_3-\eta_2;\alpha_2,\beta_2).$$

$\mathcal{P}\{A_{13}\}$ is similar to $\mathcal{P}\{A_{11}\}$, we can take the same process and change corresponding value of integrals to obtain $\mathcal{P}\{A_{13}\}$ straightly.

$$\begin{aligned}\mathcal{P}\{A_{13}\}=&\int_{\eta_2-\eta_1}^{\eta_3-\eta_2}\int_{\eta_1}^{\eta_3-\eta_1-x_3}\int_{\eta_3-x_2-x_3}^{\infty}f_{H_1|x_2,x_3}(x_1)\\&\cdot f_{H_2|x_3}(x_2)f_{H_3}(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\=&\int_{\eta_2-\eta_1}^{\eta_3-\eta_2}\int_{\eta_1}^{\eta_3-\eta_1-x_3}\frac{\beta_2^{\alpha_2}e^{-\beta_1(\eta_3-x_3)}}{\Gamma(\alpha_2)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{(-1)^{i-j}}{i!}\\&\cdot\beta_1^i(\eta_3-x_3)^jx_2^{\sigma_1-1}e^{-(\beta_2-\beta_1)x_2}\mathrm{d}x_2f_{H_3}(x_3)\mathrm{d}x_3\end{aligned}\tag{7}$$

When $\beta_1=\beta_2$, then the above expression is simply

$$\begin{aligned}\mathcal{P}\{A_{13}\}=&\int_{\eta_2-\eta_1}^{\eta_3-\eta_2}\frac{\beta_2^{\alpha_2}e^{-\beta_1(\eta_3-x_3)}}{\Gamma(\alpha_2)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i(-1)^{i-j}}{i!\sigma_1}\\&\cdot(\eta_3-x_3)^j\left((\eta_3-\eta_1-x_3)^{\sigma_1}-\eta_1^{\sigma_1}\right)f_{H_3}(x_3)\mathrm{d}x_3\\=&\mathcal{P}\{A_{13}^1\}-\mathcal{P}\{A_{13}^2\}.\end{aligned}$$

$$\begin{aligned}\mathcal{P}\{A_{13}^2\}=&\int_{\eta_2-\eta_1}^{\eta_3-\eta_2}\frac{\beta_2^{\alpha_2}e^{-\beta_1(\eta_3-x_3)}}{\Gamma(\alpha_2)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\eta_1^{\sigma_1}}{\sigma_1}\\&\cdot\frac{\beta_1^i(-1)^{i-j}(\eta_3-x_3)^j}{i!}\frac{\beta_3^{\alpha_3}x_3^{\alpha_3-1}e^{-\beta_3x_3}}{\Gamma(\alpha_3)}\mathrm{d}x_3\\=&\int_{\eta_2-\eta_1}^{\eta_3-\eta_2}\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\eta_1^{\sigma_1}}{i!\sigma_1}\\&\cdot\beta_1^i(-1)^{i-l}\eta_3^lx_3^{\sigma_2-1}e^{-(\beta_3-\beta_1)x_3}\mathrm{d}x_3,\end{aligned}$$

if $\beta'=0\,(\beta_3=\beta_1)$, we can get $\mathcal{P}\{A_{13}^2\}$ as

$$\begin{aligned}\mathcal{P}\{A_{13}^2\}=&\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i(-1)^{i-l}}{i!\sigma_1\sigma_2}\\&\cdot\eta_3^l\eta_1^{\sigma_1}\left((\eta_3-\eta_2)^{\sigma_2}-(\eta_2-\eta_1)^{\sigma_2}\right).\end{aligned}$$

otherwise $\beta'\neq0\,(\beta_3\neq\beta_1)$, we can get $\mathcal{P}\{A_{13}^2\}$ as

$$\begin{aligned}\mathcal{P}\{A_{13}^2\}=&\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i(-1)^{i-l}\Gamma(\sigma_2)}{i!\sigma_1(\beta')^{\sigma_2}}\\&\cdot\eta_3^l\eta_1^{\sigma_1}\left(F(\eta_3-\eta_2;\sigma_2,\beta')-F(\eta_2-\eta_1;\sigma_2,\beta')\right).\end{aligned}$$

For $\mathcal{P}\{A_{13}^1\}$, when $\beta_1=\beta_2$, we have

$$\begin{aligned}\mathcal{P}\{A_{13}^1\}=&\int_{\eta_2-\eta_1}^{\eta_3-\eta_2}\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i\eta_3^l}{i!\sigma_1}\\&\cdot\sum_{p=0}^{\sigma_1}\binom{\sigma_1}{p}(-1)^{\sigma_1+i-l-p}(\eta_3-\eta_1)^px_3^{\sigma_3-1}e^{-\beta'x_3}\mathrm{d}x_3.\end{aligned}$$

if $\beta'=\beta_3-\beta_1=0$, we can get $\mathcal{P}\{A_{13}^1\}$ as

$$\begin{aligned}\mathcal{P}\{A_{13}^1\}=&\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{p=0}^{\sigma_1}\binom{\sigma_1}{p}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i\eta_3^l}{i!\sigma_1\sigma_3}\\&\cdot(-1)^{\sigma_1+i-l-p}(\eta_3-\eta_1)^p\left((\eta_3-\eta_2)^{\sigma_3}-(\eta_2-\eta_1)^{\sigma_3}\right).\end{aligned}$$

otherwise $\beta'\neq0$, we can get $\mathcal{P}\{A_{13}^1\}$ as

$$\begin{aligned}\mathcal{P}\{A_{13}^1\}=&\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{p=0}^{\sigma_1}\binom{\sigma_1}{p}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i\eta_3^l\Gamma(\sigma_3)}{i!\sigma_1(\beta')^{\sigma_3}}\\&\cdot(-1)^{\sigma_1+i-l-p}(\eta_3-\eta_1)^p\\&\cdot\left(F(\eta_3-\eta_2;\sigma_3,\beta')-F(\eta_2-\eta_1;\sigma_3,\beta')\right).\end{aligned}$$

When $\beta_1$ and $\beta_2$ are not the same, then we can obtain

$$\begin{aligned}\mathcal{P}\{A_{13}\}=&\int_{\eta_2-\eta_1}^{\eta_3-\eta_2}\frac{\beta_2^{\alpha_2}e^{-\beta_1(\eta_3-x_3)}}{\Gamma(\alpha_2)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i(-1)^{i-j}}{i!(\beta_2-\beta_1)^{\sigma_1}}\\&\cdot(\eta_3-x_3)^j\Gamma(\sigma_1)\left(F'(\eta_1;\sigma_1,\beta_2-\beta_1)\right.\\&\left.-F'(\eta_3-\eta_1-x_3;\sigma_1,\beta_2-\beta_1)\right)f_{H_3}(x_3)\mathrm{d}x_3\\=&\mathcal{P}\{A_{13}^1\}-\mathcal{P}\{A_{13}^2\}.\end{aligned}\tag{8}$$

$$\begin{aligned}\mathcal{P}\{A_{13}^2\}=&\int_{\eta_2-\eta_1}^{\eta_3-\eta_2}\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i\Gamma(\sigma_1)}{i!(\beta_2-\beta_1)^{\sigma_1}}\\&\cdot\sum_{l=0}^{j}\sum_{p=0}^{\sigma_1-1}\sum_{q=0}^{p}\binom{p}{q}\binom{j}{l}\frac{(-1)^{i-l}\eta_3^le^{-(\beta_2-\beta_1)(\eta_3-\eta_1)}}{p!}\\&\cdot(\beta_2-\beta_1)^p(-1)^{p-q}(\eta_3-\eta_1)^qx_3^{\sigma_2'-1}e^{-\hat{\beta}'x_3}\mathrm{d}x_3.\end{aligned}$$

if $\hat{\beta}'=\beta_3-\beta_2=0$, $\mathcal{P}\{A_{13}^2\}$ can be calculated as

$$\begin{aligned}\mathcal{P}\{A_{13}^2\}=&\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{i}{j}\binom{j}{l}\frac{\beta_1^i\Gamma(\sigma_1)}{i!(\beta_2-\beta_1)^{\sigma_1}}\\&\cdot\sum_{p=0}^{\sigma_1-1}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^{i-l+p-q}\eta_3^le^{-(\beta_2-\beta_1)(\eta_3-\eta_1)}}{p!\sigma_2'}\\&\cdot(\beta_2-\beta_1)^p(\eta_3-\eta_1)^q\left((\eta_3-\eta_2)^{\sigma_2'}-(\eta_2-\eta_1)^{\sigma_2'}\right).\end{aligned}$$

if $\hat{\beta}'\neq0$, $\mathcal{P}\{A_{13}^2\}$ can be calculated as

$$\begin{aligned}\mathcal{P}\{A_{13}^2\}=&\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i\Gamma(\sigma_1)}{i!(\beta_2-\beta_1)^{\sigma_1}}\\&\cdot\sum_{p=0}^{\sigma_1-1}\sum_{q=0}^{p}\binom{p}{q}\frac{(-1)^{i-l+p-q}\eta_3^le^{-(\beta_2-\beta_1)(\eta_3-\eta_1)}}{p!(\hat{\beta}')^{\sigma_2'}}\\&\cdot(\beta_2-\beta_1)^p(\eta_3-\eta_1)^q\Gamma(\sigma_2')\\&\cdot\left(F\left(\eta_3-\eta_2;\sigma_2',\hat{\beta}'\right)-F\left(\eta_2-\eta_1;\sigma_2',\hat{\beta}'\right)\right).\end{aligned}$$

For $\mathcal{P}\{A_{13}^1\}$, when $\beta_1\neq\beta_2$, we have

$$\begin{aligned}\mathcal{P}\{A_{13}^1\}=&\int_{\eta_2-\eta_1}^{\eta_3-\eta_2}\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i\Gamma(\sigma_1)(-1)^{i-l}}{i!(\beta_2-\beta_1)^{\sigma_1}}\\&\cdot\sum_{l=0}^{j}\binom{j}{l}\eta_3^lF'(\eta_1;\sigma_1,\beta_2-\beta_1)\,x_3^{\sigma_2-1}e^{-\beta'x_3}\mathrm{d}x_3\end{aligned}\tag{9}$$

if $\beta' = \beta_3 - \beta_1 = 0$, we get

$$\mathcal{P}\{A_{13}^1\}=\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_2\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i(-1)^{i-l}\Gamma(\sigma_1)}{i!(\beta_2-\beta_1)^{\sigma_1}\sigma_2} \cdot\eta_3^l F'\left(\eta_1;\sigma_1,\beta_2-\beta_1\right)\left((\eta_3-\eta_2)^{\sigma_2}-(\eta_2-\eta_1)^{\sigma_2}\right).$$

otherwise $\beta' \neq 0$, we get

$$\mathcal{P}\{A_{13}^1\}=\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_2\eta_3}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_1^i(-1)^{i-l}}{i!} \cdot\frac{\Gamma(\sigma_1)\Gamma(\sigma_2)}{(\beta_2-\beta_1)^{\sigma_1}(\beta')^{\sigma_2}}\eta_3^l\left(1-F\left(\eta_1;\sigma_1,\beta_2-\beta_1\right)\right) \cdot\left(F\left(\eta_3-\eta_2;\sigma_2,\beta'\right)-F\left(\eta_2-\eta_1;\sigma_2,\beta'\right)\right).$$

According to symmetry of $\mathcal{P}\{A_{12}\}$ and $\mathcal{P}\{A_{15}\}$, we can obtain $\mathcal{P}\{A_{15}\}$ just to exchange each subscript's position which is 2 or 3 in $\mathcal{P}\{A_{12}\}$. $\mathcal{P}\{A_{14}\}$ is similar to $\mathcal{P}\{A_{12}\}$, therefore, we do not discuss them in detail, only to represent computational results. when $\beta_1$ and $\beta_2$ of $\mathcal{P}\{A_{15}\}$, $\beta_2$ and $\beta_3$ of $\mathcal{P}\{A_{14}\}$ are respectively the same, the expression can be obtained as

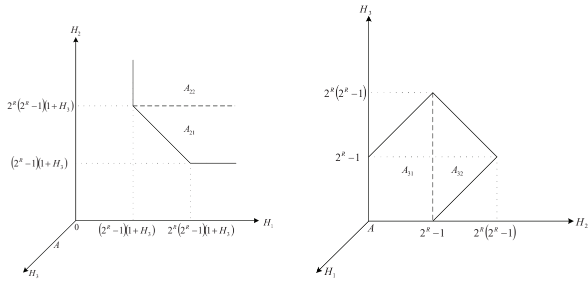
$$\begin{aligned}\mathcal{P}\{A_{14}\}=&\int_{\eta_2-\eta_1}^{\eta_3-\eta_2}\int_{\eta_3-\eta_1-x_3}^{\infty}\int_{\eta_1}^{\infty}f_{H_1|x_2,x_3}(x_1) \cdot f_{H_2|x_3}(x_2)f_{H_3}(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\ =&\frac{\beta_3^{\alpha_3}e^{-\beta_2(\eta_3-\eta_1)}}{\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_2^i(-1)^{i-j}(\eta_3-\eta_1)^j}{i!\grave{\sigma}_1} \cdot\left((\eta_3-\eta_2)^{\grave{\sigma}_1}-(\eta_2-\eta_1)^{\grave{\sigma}_1}\right)\left(1-F\left(\eta_1;\alpha_1,\beta_1\right)\right),\end{aligned}\quad(10)$$

$$\begin{aligned}\mathcal{P}\{A_{15}\}=&\int_{\eta_3-\eta_2}^{\infty}\int_{\eta_1}^{\eta_2-\eta_1}\int_{\eta_2-x_3}^{\infty}f_{H_1|x_2,x_3}(x_1) \cdot f_{H_2|x_3}(x_2)f_{H_3}(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\ =&\frac{\beta_2^{\alpha_2}e^{-\beta_1\eta_2}}{\Gamma(\alpha_2)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i(-1)^{i-j}\eta_2^j}{i!\sigma_1} \cdot\left((\eta_2-\eta_1)^{\sigma_1}-\eta_1^{\sigma_1}\right)\left(1-F\left(\eta_3-\eta_2;\alpha_3,\beta_3\right)\right)\end{aligned}\quad(11)$$

when $\beta_1$ and $\beta_2$ of $\mathcal{P}\{A_{15}\}$, $\beta_2$ and $\beta_3$ of $\mathcal{P}\{A_{14}\}$ are not equivalent to each other, then we can obtain

$$\mathcal{P}\{A_{14}\}=\frac{\beta_3^{\alpha_3}e^{-\beta_2(\eta_3-\eta_1)}}{\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_2^i(-1)^{i-j}(\eta_3-\eta_1)^j\Gamma(\grave{\sigma}_1)}{i!(\hat{\beta}')^{\grave{\sigma}_1}} \cdot\left(F\left(\eta_3-\eta_2;\grave{\sigma}_1,\hat{\beta}'\right)-F\left(\eta_2-\eta_1;\grave{\sigma}_1,\hat{\beta}'\right)\right)F'(\eta_1;\alpha_1,\beta_1),$$

$$\mathcal{P}\{A_{15}\}=\frac{\beta_2^{\alpha_2}e^{-\beta_1\eta_2}}{\Gamma(\alpha_2)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_1^i(-1)^{i-j}\eta_2^j\Gamma(\sigma_1)}{i!(\beta_2-\beta_1)^{\sigma_1}} \cdot\left(F\left(\eta_2-\eta_1;\sigma_1,\beta_2-\beta_1\right)-F\left(\eta_1;\sigma_1,\beta_2-\beta_1\right)\right) \cdot\left(1-F\left(\eta_3-\eta_2;\alpha_3,\beta_3\right)\right),$$




(a) rate regions of $\tilde{P}_{12\bar{3}}$ (b) rate regions of $\tilde{P}_{1\bar{2}\bar{3}}$

Fig. 2. Rate regions of 3-user MAC channel

Finally, $\mathcal{P}\{A_{16}\}$ is a simple integral as

$$\begin{aligned}\mathcal{P}\{A_{16}\}=&\int_{\eta_3-\eta_2}^{\infty}\int_{\eta_2-\eta_1}^{\infty}\int_{\eta_1}^{\infty}f_{H_1|x_2,x_3}(x_1) \cdot f_{H_2|x_3}(x_2)f_{H_3}(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\ =&\left(1-F\left(\eta_1;\alpha_1,\beta_1\right)\right)\left(1-F\left(\eta_2-\eta_1;\alpha_2,\beta_2\right)\right) \cdot\left(1-F\left(\eta_3-\eta_2;\alpha_3,\beta_3\right)\right).\end{aligned}\quad(12)$$

As a result, we obtanin $\tilde{P}_{123}$ as

$$\tilde{P}_{123} = \mathcal{P}\{A_{11}\}+\mathcal{P}\{A_{12}\}+\mathcal{P}\{A_{13}\} +\mathcal{P}\{A_{14}\}+\mathcal{P}\{A_{15}\}+\mathcal{P}\{A_{16}\}\quad(13)$$

The probability of $b$ can decode $a_1$ and $a_2$ but not $a_3$ is denoted as $\tilde{P}_{12\bar{3}} = \Pr\{E_{1,\mathcal{A}}, E_{2,\bar{\mathcal{A}}} : \mathcal{A} = \{a_1, a_2\}, \mathcal{L} = \{a_1, a_2, a_3\}\}$, i.e.,

$$\begin{aligned}\tilde{P}_{12\bar{3}} =& \Pr\Big\{\log_2\left(1+\frac{H_1}{1+H_3}\right)>R, \log_2\left(1+\frac{H_2}{1+H_3}\right)>R,\\ &\log_2\left(1+\frac{H_1+H_2}{1+H_3}\right)>2R, \log_2\left(1+H_3\right)<R\Big\}\\ =&\mathcal{P}\left\{\frac{H_1}{1+H_3}>2^R-1, \frac{H_2}{1+H_3}>2^R-1, \frac{H_1+H_2}{1+H_3}>2^{2R}-1, H_3<2^R-1\right\},\end{aligned}$$

To derive the probability, we need to calculate two independent subregions $A_{21}$ and $A_{22}$ , from Fig. 2 (a) we know that the integral regions are three-dimensional, axis $H_1$ and $H_2$ consist of the plane, the height is between 0 and point $A(H_1, H_2, \eta_1)$, therefore

$$\begin{aligned}\mathcal{P}\{A_{21}\}=&\int_0^{\eta_1}\int_{\eta_1(1+x_3)}^{(\eta_2-\eta_1)(1+x_3)}\int_{\eta_2(1+x_3)-x_2}^{\infty}f_{H_1|x_2,x_3}(x_1)\\&\cdot f_{H_2|x_3}(x_2)f_{H_3}(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\=&\int_0^{\eta_1}\int_{\eta_1(1+x_3)}^{(\eta_2-\eta_1)(1+x_3)}\int_{\eta_2(1+x_3)-x_1}^{\infty}f_{H_2|x_1,x_3}(x_2)\\&\cdot f_{H_1|x_3}(x_1)f_{H_3}(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\=&\int_0^{\eta_1}\int_{\eta_1(1+x_3)}^{(\eta_2-\eta_1)(1+x_3)}(1-F\left(\eta_2(1+x_3)-x_1;\alpha_2,\beta_2\right))\\&\cdot\frac{\beta_1^{\alpha_1}x_1^{\alpha_1-1}e^{-\beta_1x_1}}{\Gamma(\alpha_1)}\mathrm{d}x_1f_{H_3}(x_3)\mathrm{d}x_3\\=&\int_0^{\eta_1}\int_{\eta_1(1+x_3)}^{(\eta_2-\eta_1)(1+x_3)}\frac{\beta_1^{\alpha_1}e^{-\beta_2\eta_2(1+x_3)}}{\Gamma(\alpha_1)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\binom{i}{j}\\&\cdot\frac{\beta_2^i(-1)^{i-j}(\eta_2(1+x_3))^j}{i!}\\&\cdot x_1^{\acute{\sigma_1}-1}e^{-(\beta_1-\beta_2)x_1}\mathrm{d}x_1f_{H_3}(x_3)\mathrm{d}x_3.\end{aligned}\tag{14}$$

in which $\acute{\sigma_1}=\alpha_1+i-j$. When $\beta_1-\beta_2=0$, it is simple to obtain that

$$\begin{aligned}\mathcal{P}\{A_{21}\}=&\int_0^{\eta_1}\frac{\beta_1^{\alpha_1}e^{-\beta_2\eta_2(1+x_3)}}{\Gamma(\alpha_1)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_2^i(-1)^{i-j}}{i!\acute{\sigma_1}}\\&\cdot(\eta_2(1+x_3))^j\left(((\eta_2-\eta_1)(1+x_3))^{\acute{\sigma_1}}\right.\\&\left.-(\eta_1(1+x_3))^{\acute{\sigma_1}}\right)\frac{\beta_3^{\alpha_3}x_3^{\alpha_3-1}e^{-\beta_3x_3}}{\Gamma(\alpha_3)}\mathrm{d}x_3\\=&\mathcal{P}\{A_{21}^1\}-\mathcal{P}\{A_{21}^2\}.\end{aligned}\tag{15}$$

To simply represent $\mathcal{P}\{A_{21}^1\}$ and $\mathcal{P}\{A_{21}^2\}$, set $\acute{\sigma_3}=\alpha_1+\alpha_3+i-l-p$ and $\grave{\beta}=\beta_3+\beta_2(2^{2R}-1)$, then we obtain that

$$\begin{aligned}\mathcal{P}\{A_{21}^1\}=&\int_0^{\eta_1}\frac{\beta_1^{\alpha_1}\beta_3^{\alpha_3}e^{-\beta_2\eta_2}}{\Gamma(\alpha_1)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{p=0}^{\acute{\sigma_1}}\binom{\acute{\sigma_1}}{p}\binom{j}{l}\binom{i}{j}\\&\cdot\frac{\beta_2^i(-1)^{i-j}}{i!\acute{\sigma_1}}\eta_2^j(\eta_2-\eta_1)^{\acute{\sigma_1}}x_3^{\acute{\sigma_3}-1}e^{-\grave{\beta}x_3}\mathrm{d}x_3\\=&\frac{\beta_1^{\alpha_1}\beta_3^{\alpha_3}e^{-\beta_2\eta_2}}{\Gamma(\alpha_1)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{p=0}^{\acute{\sigma_1}}\binom{\acute{\sigma_1}}{p}\binom{j}{l}\binom{i}{j}\\&\cdot\frac{\beta_2^i(-1)^{i-j}\Gamma(\acute{\sigma_3})}{i!\acute{\sigma_1}(\grave{\beta})^{\acute{\sigma_3}}}\eta_2^j(\eta_2-\eta_1)^{\acute{\sigma_1}}F\left(\eta_1;\acute{\sigma_3},\grave{\beta}\right),\end{aligned}$$

$$\begin{aligned}\mathcal{P}\{A_{21}^2\}=&\int_0^{\eta_1}\frac{\beta_1^{\alpha_1}\beta_3^{\alpha_3}e^{-\beta_2\eta_2}}{\Gamma(\alpha_1)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{p=0}^{\acute{\sigma_1}}\binom{\acute{\sigma_1}}{p}\binom{j}{l}\binom{i}{j}\\&\cdot\frac{\beta_2^i(-1)^{i-j}}{i!\acute{\sigma_1}}\eta_2^j\eta_1^{\acute{\sigma_1}}x_3^{\acute{\sigma_3}-1}e^{-\grave{\beta}x_3}\mathrm{d}x_3\\=&\frac{\beta_1^{\alpha_1}\beta_3^{\alpha_3}e^{-\beta_2\eta_2}}{\Gamma(\alpha_1)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\sum_{p=0}^{\acute{\sigma_1}}\binom{\acute{\sigma_1}}{p}\binom{j}{l}\binom{i}{j}\\&\cdot\frac{\beta_2^i(-1)^{i-j}\Gamma(\acute{\sigma_3})}{i!\acute{\sigma_1}(\grave{\beta})^{\acute{\sigma_3}}}\eta_2^j\eta_1^{\alpha'}F\left(\eta_1;\acute{\sigma_3},\grave{\beta}\right).\end{aligned}$$

When $\beta_1-\beta_2\neq0$, we obtain that

$$\begin{aligned}\mathcal{P}\{A_{21}\}=&\int_0^{\eta_1}\frac{\beta_1^{\alpha_1}e^{-\beta_2\eta_2(1+x_3)}}{\Gamma(\alpha_1)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\Gamma(\acute{\sigma_1})\beta_2^i(-1)^{i-j}}{i!(\beta_1-\beta_2)^{\acute{\sigma_1}}}\\&\cdot(\eta_2(1+x_3))^j\left(F'\left(\eta_1(1+x_3);\acute{\sigma_1},\beta_1-\beta_2\right)\right.\\&\left.-F'((\eta_2-\eta_1)(1+x_3);\acute{\sigma_1},\beta_1-\beta_2)\right)\frac{\beta_3^{\alpha_3}x_3^{\alpha_3-1}e^{-\beta_3x_3}}{\Gamma(\alpha_3)}\mathrm{d}x_3\\=&\mathcal{P}\{A_{21}^1\}-\mathcal{P}\{A_{21}^2\}.\end{aligned}\tag{16}$$

To simply represent $\mathcal{P}\{A_{21}^1\}$ and $\mathcal{P}\{A_{21}^2\}$ , set $\sigma_2'=\alpha_3+j-l+p-q$ and $\bar{\beta}=\beta_3+\beta_2(2^{2R}-2^R)+\beta_1(2^R-1)$, $\tilde{\beta}=\beta_3+\beta_1(2^{2R}-2^R)+\beta_2(2^R-1)$, then we obtain that

$$\begin{aligned}\mathcal{P}\{A_{21}^1\}=&\int_0^{\eta_1}\frac{\beta_1^{\alpha_1}\beta_3^{\alpha_3}e^{-\beta_2\eta_2}}{\Gamma(\alpha_1)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_2^i(-1)^{i-j}}{i!}\\&\cdot\frac{\eta_2^j\Gamma(\acute{\sigma_1})}{(\beta_1-\beta_2)^{\acute{\sigma_1}}}\sum_{p=0}^{\acute{\sigma_1}-1}\sum_{q=0}^{p}\binom{p}{q}\frac{((\beta_1-\beta_2)\eta_1)^p}{p!}\\&\cdot e^{-(\beta_1-\beta_2)\eta_1}x_3^{\acute{\sigma_2}-1}e^{-\bar{\beta}x_3}\mathrm{d}x_3\\=&\frac{\beta_1^{\alpha_1}\beta_3^{\alpha_3}e^{-\beta_2\eta_2}}{\Gamma(\alpha_1)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_2^i(-1)^{i-j}}{i!}\\&\cdot\frac{\eta_2^j\Gamma(\acute{\sigma_1})}{(\beta_1-\beta_2)^{\acute{\sigma_1}}}\sum_{p=0}^{\acute{\sigma_1}-1}\sum_{q=0}^{p}\binom{p}{q}\frac{((\beta_1-\beta_2)\eta_1)^p}{p!(\bar{\beta})^{\acute{\sigma_2}}}\\&\cdot e^{-(\beta_1-\beta_2)\eta_1}\Gamma(\acute{\sigma_2})F\left(\eta_1;\acute{\sigma_2},\bar{\beta}\right),\end{aligned}$$

$$\begin{aligned}\mathcal{P}\{A_{21}^2\}=&\int_0^{\eta_1}\frac{\beta_1^{\alpha_1}\beta_3^{\alpha_3}e^{-\beta_2\eta_2}}{\Gamma(\alpha_1)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_2^i(-1)^{i-j}}{i!}\\&\cdot\frac{\eta_2^j\Gamma(\acute{\sigma_1})}{(\beta_1-\beta_2)^{\acute{\sigma_1}}}\sum_{p=0}^{\acute{\sigma_1}-1}\sum_{q=0}^{p}\binom{p}{q}\frac{((\beta_1-\beta_2)(\eta_2-\eta_1))^p}{p!}\\&\cdot e^{-(\beta_1-\beta_2)(\eta_2-\eta_1)}x_3^{\acute{\sigma_2}-1}e^{-\tilde{\beta}x_3}\mathrm{d}x_3\\=&\frac{\beta_1^{\alpha_1}\beta_3^{\alpha_3}e^{-\beta_2\eta_2}}{\Gamma(\alpha_1)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{\beta_2^i(-1)^{i-j}}{i!}\\&\cdot\frac{\eta_2^j\Gamma(\acute{\sigma_1})}{(\beta_1-\beta_2)^{\acute{\sigma_1}}}\sum_{p=0}^{\acute{\sigma_1}-1}\sum_{q=0}^{p}\binom{p}{q}\frac{((\beta_1-\beta_2)(\eta_2-\eta_1))^p\Gamma(\acute{\sigma_2})}{p!(\tilde{\beta})^{\acute{\sigma_2}}}\\&\cdot e^{-(\beta_1-\beta_2)(\eta_2-\eta_1)}F\left(\eta_1;\acute{\sigma_2},\tilde{\beta}\right),\end{aligned}$$

To obtain $\mathcal{P}\{A_{22}\}$, set $\sigma_2'=\alpha_3+i-j+p-q$ and $\tilde{\beta}=$

$\beta_3+\beta_2(2^R-1)+\beta_1(2^{2R}-2^R)$, then we can get

$$
\begin{aligned}
\mathcal{P}\{A_{22}\}=&\int_0^{\eta_1}\int_{(\eta_2-\eta_1)(1+x_3)}^{\infty}\int_{\eta_1(1+x_3)}^{\infty}f_{H_1|x_2,x_3}(x_1)\\
&\quad\cdot f_{H_2|x_3}(x_2)f_{H_3}(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
=&\int_0^{\eta_1}\int_{(\eta_2-\eta_1)(1+x_3)}^{\infty}\int_{\eta_1(1+x_3)}^{\infty}f_{H_2|x_1,x_3}(x_2)\\
&\quad\cdot f_{H_1|x_3}(x_1)f_{H_3}(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
=&\int_0^{\eta_1}\left(1-F\left(\eta_1(1+x_3);\alpha_2,\beta_2\right)\right)\\
&\cdot\left(1-F\left((\eta_2-\eta_1)(1+x_3);\alpha_1,\beta_1\right)\right)f_{H_3}(x_3)\mathrm{d}x_3\\
=&\int_0^{\eta_1}\frac{\beta_3^{\alpha_3}e^{-\beta_1(\eta_2-\eta_1)}e^{-\beta_2\eta_1}}{\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\binom{i}{j}\frac{(\beta_2\eta_1)^i}{i!}\\
&\cdot\sum_{p=0}^{\alpha_1-1}\sum_{q=0}^{p}\binom{p}{q}\frac{(\beta_1(\eta_2-\eta_1))^p}{p!}x_3^{\acute{\sigma}_2-1}e^{-\tilde{\beta}x_3}\mathrm{d}x_3\\
=&\frac{\beta_3^{\alpha_3}e^{-\beta_1(\eta_2-\eta_1)}e^{-\beta_2\eta_1}}{\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_2-1}\sum_{j=0}^{i}\binom{i}{j}\frac{(\beta_2\eta_1)^i}{i!}\\
&\cdot\sum_{p=0}^{\alpha_1-1}\sum_{q=0}^{p}\binom{p}{q}\frac{(\beta_1(\eta_2-\eta_1))^p\Gamma(\acute{\sigma}_2)}{p!(\tilde{\beta})^{\acute{\sigma}_2}}F\left(\eta_1;\acute{\sigma}_2,\tilde{\beta}\right)
\end{aligned}
\tag{17}
$$

Finally, from $\tilde{P}_{12\bar{3}}=\mathcal{P}\{A_{21}\}+\mathcal{P}\{A_{22}\}$, the probability can be derived. Due to symmetry, we can derive $\tilde{P}_{1\bar{2}3}$ and $\tilde{P}_{\bar{1}23}$ from $\tilde{P}_{12\bar{3}}$ only to exchange subscript of parameters. Then the probability of $b$ can decode $a_1$ and but not $a_2$, $a_3$ is denoted as $\tilde{P}_{1\bar{2}\bar{3}}=\Pr\{E_{1,\mathcal{A}},E_{2,\bar{\mathcal{A}}}:\mathcal{A}=\{a_1\},\mathcal{L}=\{a_1,a_2,a_3\}\}$, this is also need to calculate two subregions $A_{31}$ and $A_{32}$, the rate region is as Fig. 2 (b), coordinate of point $A$ is $(0,0,\eta_1(1+H_2+H_3))$, this means integral plane is consist of $H_2$ and $H_3$, the height is larger than $\eta_1(1+H_2+H_3)$, i.e.,

$$
\begin{aligned}
\tilde{P}_{1\bar{2}\bar{3}}=&\Pr\Big\{\log_2\left(1+\frac{H_1}{1+H_2+H_3}\right)>R,\ \log_2\left(1+\frac{H_2}{1+H_3}\right)<R,\\
&\log_2\left(1+\frac{H_3}{1+H_2}\right)<R,\ \log_2(1+H_2+H_3)<2R\Big\}\\
=&\Pr\left\{\frac{H_1}{1+H_2+H_3}>2^R-1,\ \frac{H_2}{1+H_3}<2^R-1,\right.\\
&\left.\frac{H_3}{1+H_2}<2^R-1,\ H_2+H_3<2^{2R}-1\right\}.
\end{aligned}
\tag{18}
$$

Set that $\grave{\sigma}_1=\alpha_3+i-j$ and $\acute{\beta}=\beta_3+\beta_1(2^R-1)$, $\mathcal{P}\{A_{31}\}$ can be obtained as

$$
\begin{aligned}
\mathcal{P}\{A_{31}\}=&\int_0^{\eta_1}\int_0^{\eta_1(1+x_3)}\int_{\eta_1(1+x_2+x_3)}^{\infty}f_{H_1|x_2,x_3}(x_1)\\
&\quad\cdot f_{H_2|x_3}(x_2)f_{H_3}(x_3)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
=&\int_0^{\eta_1}\int_0^{\eta_1(1+x_2)}\int_{\eta_1(1+x_2+x_3)}^{\infty}f_{H_1|x_3,x_2}(x_1)\\
&\quad\cdot f_{H_3|x_2}(x_3)f_{H_2}(x_2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\\
=&\int_0^{\eta_1}\int_0^{\eta_1(1+x_2)}F'\left(\eta_1(1+x_2+x_3);\alpha_1,\beta_1\right)\\
&\quad\cdot\frac{\beta_3^{\alpha_3}x_3^{\alpha_3-1}e^{-\beta_3x_3}}{\Gamma(\alpha_3)}f_{H_2}(x_2)\mathrm{d}x_2\mathrm{d}x_3\\
=&\int_0^{\eta_1}\int_0^{\eta_1(1+x_2)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_3^{\alpha_3}e^{-\beta_1\eta_1(1+x_2)}}{i!\Gamma(\alpha_3)}\\
&\cdot(\beta_1\eta_1)^i(1+x_2)^jx_3^{\grave{\sigma}_1-1}e^{-\beta'x_3}\mathrm{d}x_3f_{H_2}(x_2)\mathrm{d}x_2\\
=&\int_0^{\eta_1}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\binom{i}{j}\frac{\beta_3^{\alpha_3}e^{-\beta_1\eta_1(1+x_2)}\Gamma(\grave{\sigma}_1)}{i!\Gamma(\alpha_3)(\acute{\beta})^{\grave{\sigma}_1}}\\
&\cdot(\beta_1\eta_1)^i(1+x_2)^jF\left(\eta_1(1+x_2);\grave{\sigma}_1,\acute{\beta}\right)f_{H_2}(x_2)\mathrm{d}x_2\\
=&\mathcal{P}\{A_{31}^1\}-\mathcal{P}\{A_{31}^2\}
\end{aligned}
\tag{19}
$$

For simplicity, set $\acute{\sigma}_2=\alpha_2+j-l$ and $\dot{\beta}=\beta_2+\beta_1\eta_1$, then $\mathcal{P}\{A_{31}^1\}$ can be obtained as

$$
\begin{aligned}
\mathcal{P}\{A_{31}^1\}=&\int_0^{\eta_1}\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_1}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}\binom{j}{l}\binom{i}{j}\frac{(\beta_1\eta_1)^i}{i!(\acute{\beta})^{\grave{\sigma}_1}}\\
&\cdot\Gamma(\grave{\sigma}_1)x_2^{\acute{\sigma}_2-1}e^{-\dot{\beta}x_2}\mathrm{d}x_2\\
=&\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_1}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{j=0}^{l}\binom{j}{l}\binom{i}{j}\frac{(\beta_1\eta_1)^i}{i!(\acute{\beta})^{\grave{\sigma}_1}(\dot{\beta})^{\acute{\sigma}_2}}\\
&\cdot\Gamma(\grave{\sigma}_1)\Gamma(\acute{\sigma}_2)F\left(\eta_1;\acute{\sigma}_2,\dot{\beta}\right),
\end{aligned}
$$

For simplicity, set $\acute{\sigma}_2'=\alpha_2+j-l+p-q$ and $\ddot{\beta}=\beta_3(2^R-1)+\beta_2+\beta_1(2^{2R}-2^R)$, then $\mathcal{P}\{A_{31}^2\}$ can be obtained as

$$
\begin{aligned}
\mathcal{P}\{A_{31}^2\}=&\int_0^{\eta_1}\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_1}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{l=0}^{j}!\binom{j}{l}\binom{i}{j}\frac{(\beta_1\eta_1)^i}{i!(\acute{\beta})^{\grave{\sigma}_1}}\\
&\cdot\Gamma(\grave{\sigma}_1)\sum_{p=0}^{\grave{\sigma}_1-1}\sum_{q=0}^{p}\binom{p}{q}\frac{(\acute{\beta}\eta_1)^p}{p!}e^{-\acute{\beta}\eta_1}x_2^{\acute{\sigma}_2'-1}e^{-\ddot{\beta}x_2}\mathrm{d}x_2\\
=&\frac{\beta_2^{\alpha_2}\beta_3^{\alpha_3}e^{-\beta_1\eta_1}}{\Gamma(\alpha_2)\Gamma(\alpha_3)}\sum_{i=0}^{\alpha_1-1}\sum_{j=0}^{i}\sum_{j=0}^{l}\binom{j}{l}\binom{i}{j}\frac{(\beta_1\eta_1)^i\Gamma(\grave{\sigma}_1)}{i!(\acute{\beta})^{\grave{\sigma}_1}}\\
&\cdot\sum_{p=0}^{\grave{\sigma}_1-1}\sum_{q=0}^{p}\binom{p}{q}\frac{(\acute{\beta}\eta_1)^p\Gamma(\acute{\sigma}_2')}{p!(\ddot{\beta})^{\acute{\sigma}_2'}}e^{-\acute{\beta}\eta_1}F\left(\eta_1;\acute{\sigma}_2',\ddot{\beta}\right),
\end{aligned}
$$

The first step integral of $x_1$ is the same, set $\grave{\sigma}_1=\alpha_3+i-j$

and $\acute{\beta} = \beta_3 + \beta_1(2^R - 1)$. we can directly have

$$
\begin{aligned}
\mathcal{P}\{A_{32}\} =& \int_{\eta_1}^{\eta_2-\eta_1} \int_{\frac{x_3}{\eta_1}-1}^{\eta_2-x_3} \int_{\eta_1(1+x_2+x_3)}^{\infty} f_{H_1|x_2,x_3}(x_1) \\
& \cdot f_{H_2|x_3}(x_2) f_{H_3}(x_3) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \\
=& \int_{\eta_1}^{\eta_2-\eta_1} \int_{\frac{x_2}{\eta_1}-1}^{\eta_2-x_2} \int_{\eta_1(1+x_2+x_3)}^{\infty} f_{H_1|x_3,x_2}(x_1) \\
& \cdot f_{H_3|x_2}(x_3) f_{H_2}(x_2) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \\
=& \int_{\eta_1}^{\eta_2-\eta_1} \int_{\frac{x_2}{\eta_1}-1}^{\eta_2-x_2} F'\left(\eta_1(1+x_2+x_3); \alpha_1, \beta_1\right) \\
& \cdot \frac{\beta_3^{\alpha_3} x_3^{\alpha_3-1} e^{-\beta_3 x_3}}{\Gamma(\alpha_3)} f_{H_2}(x_2) \mathrm{d}x_2 \mathrm{d}x_3 \\
=& \int_{\eta_1}^{\eta_2-\eta_1} \int_{\frac{x_2}{\eta_1}-1}^{\eta_2-x_2} \sum_{i=0}^{\alpha_1-1} \sum_{j=0}^{i} \binom{i}{j} \frac{\beta_3^{\alpha_3} e^{-\beta_1\eta_1(1+x_2)}}{i!\Gamma(\alpha_3)} \\
& \cdot (\beta_1\eta_1)^i (1+x_2)^j x_3^{\grave{\sigma}_1-1} e^{-\acute{\beta} x_3} f_{H_2}(x_2) \mathrm{d}x_2 \mathrm{d}x_3 \\
=& \int_{\eta_1}^{\eta_2-\eta_1} \sum_{i=0}^{\alpha_1-1} \sum_{j=0}^{i} \binom{i}{j} \frac{\beta_3^{\alpha_3} e^{-\beta_1\eta_1(1+x_2)}}{i!\Gamma(\alpha_3)} \\
& \cdot (\beta_1\eta_1)^i (1+x_2)^j \frac{\Gamma(\grave{\sigma}_1)}{(\acute{\beta})^{\grave{\sigma}_1}} \left( F'\left(\frac{x_2}{\eta_1}-1; \grave{\sigma}_1, \acute{\beta}\right) \right. \\
& \left. - F'\left(\eta_2 - x_2; \grave{\sigma}_1, \acute{\beta}\right) \right) f_{H_2}(x_2) \mathrm{d}x_2 \\
=& \mathcal{P}\{A_{32}^1\} - \mathcal{P}\{A_{32}^2\}
\end{aligned}
\quad (20)
$$

Set $\acute{\sigma_2}' = \alpha_2 + j - l + p - q$ and $\breve{\beta} = \frac{\beta_3}{\eta_1} + \beta_2 + \beta_1(\eta_1 + 1)$, we can get $\mathcal{P}\{A_{32}^1\}$ as

$$
\begin{aligned}
\mathcal{P}\{A_{32}^1\} =& \int_{\eta_1}^{\eta_2-\eta_1} \sum_{i=0}^{\alpha_1-1} \sum_{j=0}^{i} \sum_{l=0}^{j} \binom{j}{l} \binom{i}{j} \frac{\beta_2^{\alpha_2} \beta_3^{\alpha_3} e^{-\beta_1\eta_1} e^{\acute{\beta}}}{\Gamma(\alpha_2)\Gamma(\alpha_3)} \\
& \cdot \frac{(\beta_1\eta_1)^i}{i!} \frac{\Gamma(\grave{\sigma}_1)}{(\acute{\beta})^{\grave{\sigma}_1}} \sum_{p=0}^{\grave{\sigma}_1-1} \sum_{q=0}^{p} \binom{p}{q} \frac{(\acute{\beta})^p (\frac{1}{\eta_1})^{p-q} (-1)^q}{p!} \\
& \cdot x_2^{\acute{\sigma_2}'-1} e^{-\breve{\beta} x_2} \mathrm{d}x_2 \\
=& \frac{\beta_2^{\alpha_2} \beta_3^{\alpha_3} e^{-\beta_1\eta_1} e^{\acute{\beta}}}{\Gamma(\alpha_2)\Gamma(\alpha_3)} \sum_{i=0}^{\alpha_1-1} \sum_{j=0}^{i} \sum_{l=0}^{j} \binom{j}{l} \binom{i}{j} \frac{(\beta_1\eta_1)^i}{i!} \\
& \cdot \frac{\Gamma(\grave{\sigma}_1)}{(\acute{\beta})^{\grave{\sigma}_1}} \sum_{p=0}^{\grave{\sigma}_1-1} \sum_{q=0}^{p} \binom{p}{q} \frac{(\acute{\beta})^p (\frac{1}{\eta_1})^{p-q} (-1)^q}{p!} \\
& \cdot \frac{\Gamma(\acute{\sigma_2}')}{(\breve{\beta})^{\acute{\sigma_2}'}} \left( F\left(\eta_2 - \eta_1; \acute{\sigma_2}', \breve{\beta}\right) - F\left(\eta_1; \acute{\sigma_2}', \breve{\beta}\right) \right),
\end{aligned}
$$

Set $\hat{\beta}_1' = \beta_2 - \beta_3$, we can get $\mathcal{P}\{A_{32}^2\}$ as

$$
\begin{aligned}
\mathcal{P}\{A_{32}^2\} =& \int_{\eta_1}^{\eta_2-\eta_1} \sum_{i=0}^{\alpha_1-1} \sum_{j=0}^{i} \binom{i}{j} \frac{\beta_2^{\alpha_2} \beta_3^{\alpha_3} e^{-\beta_1\eta_1} e^{-\acute{\beta}\eta_2} (\beta_1\eta_1)^i}{i!\Gamma(\alpha_2)\Gamma(\alpha_3)} \\
& \cdot \sum_{l=0}^{j} \binom{j}{l} \frac{\Gamma(\grave{\sigma}_1)}{(\acute{\beta})^{\grave{\sigma}_1}} \sum_{p=0}^{\grave{\sigma}_1-1} \sum_{q=0}^{p} \binom{p}{q} \frac{(\acute{\beta})^p (-1)^{p-q} \eta_2^q}{p!} \\
& \cdot x_2^{\acute{\sigma_2}'-1} e^{-\hat{\beta}_1' x_2} \mathrm{d}x_2.
\end{aligned}
\quad (21)
$$

If $\beta_2 - \beta_3 = 0$, it is simple to obtain

$$
\begin{aligned}
\mathcal{P}\{A_{32}^2\} =& \sum_{i=0}^{\alpha_1-1} \sum_{j=0}^{i} \sum_{l=0}^{j} \binom{j}{l} \binom{i}{j} \frac{\beta_2^{\alpha_2} \beta_3^{\alpha_3} e^{-\beta_1\eta_1} e^{-\acute{\beta}\eta_2} (\beta_1\eta_1)^i}{i!\Gamma(\alpha_2)\Gamma(\alpha_3)} \\
& \cdot \frac{\Gamma(\grave{\sigma}_1)}{(\acute{\beta})^{\grave{\sigma}_1}} \sum_{p=0}^{\grave{\sigma}_1-1} \sum_{q=0}^{p} \binom{p}{q} \frac{(\acute{\beta})^p (-1)^{p-q} \eta_2^q}{p! \acute{\sigma_2}'} \\
& \left( (\eta_2 - \eta_1)^{\acute{\sigma_2}'} - (\eta_1)^{\acute{\sigma_2}'} \right),
\end{aligned}
$$

If $\beta_2 - \beta_3 \neq 0$, we obtain

$$
\begin{aligned}
\mathcal{P}\{A_{32}^2\} =& \sum_{i=0}^{\alpha_1-1} \sum_{j=0}^{i} \sum_{l=0}^{j} \binom{j}{l} \binom{i}{j} \frac{\beta_2^{\alpha_2} \beta_3^{\alpha_3} e^{-\beta_1\eta_1} e^{-\acute{\beta}\eta_2} (\beta_1\eta_1)^i}{i!\Gamma(\alpha_2)\Gamma(\alpha_3)} \\
& \cdot \frac{\Gamma(\grave{\sigma}_1)}{(\acute{\beta})^{\grave{\sigma}_1}} \sum_{p=0}^{\grave{\sigma}_1-1} \sum_{q=0}^{p} \binom{p}{q} \frac{(\acute{\beta})^p (-1)^{p-q} \eta_2^q \Gamma(\acute{\sigma_2}')}{p! (\hat{\beta}_1')^{\acute{\sigma_2}'}} \\
& \cdot \left( F\left(\eta_2 - \eta_1; \acute{\sigma_2}', \hat{\beta}_1'\right) - F\left(\eta_1; \acute{\sigma_2}', \hat{\beta}_1'\right) \right),
\end{aligned}
$$

Furthermore, $\tilde{P}_{\bar{1}2\bar{3}}$ and $\tilde{P}_{\bar{1}\bar{2}3}$ can be calculated similarly as above procedure, we can obtain the results only to exchange some parameters' subscript, we do dot discuss in detail anymore. Finally, the probability of $b$ can decode none of them is denoted as$\tilde{P}_{\bar{1}\bar{2}\bar{3}}$, we can obtain it simply as follows.

$$
\tilde{P}_{\bar{1}\bar{2}\bar{3}} = 1 - \tilde{P}_{123} - \tilde{P}_{12\bar{3}} - \tilde{P}_{1\bar{2}3} - \tilde{P}_{\bar{1}23} - \tilde{P}_{1\bar{2}\bar{3}} - \tilde{P}_{\bar{1}2\bar{3}} - \tilde{P}_{\bar{1}\bar{2}3} \quad (22)
$$

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